AN INTRODUCTION TO LIE ALGEBRAS AND LIE GROUPS

Contents

1. Lie Algebras

1.1. General Algebras.

Definition 1.1.
1. An *algebra* over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space A equipped with a bilinear map
$$A \times A \to A, \qquad (a,b) \mapsto a.b.$$
2. An *associative algebra* is an algebra $(A, \cdot)$ such that
$$a.(b.c) = (a.b).c, \qquad \text{for all } a,b,c \in A.$$
3. A *unital* algebra is an algebra $(A, \cdot)$ containing an element 1 satisfying
$$1.a = a.1 = a, \qquad \text{for all } a \in A.$$
4. An algebra $(A, .)$ is called *commutative* if
$$a.b = b.a, \qquad \text{for all } a,b \in A.$$
5. A $\mathbb{K}$-linear map $f : A \to B$ between two $\mathbb{K}$-algebras is called an *algebra homomorphism* if
$$f(a.b) = f(a).f(b), \qquad \text{for all } a,b \in A.$$
It is called an algebra isomorphism if f is bijective.
6. An algebra homomorphism $f : A \to B$, between two unital algebras A, B, is said to be *unital* if $f(1_A) = 1_B$.

Example 1.2. The field of real numbers $\mathbb{R}$ is an algebra over itself. The field of complex numbers $\mathbb{C}$ is an algebra over itself and over $\mathbb{R}$. Both are commutative.

Example 1.3. The quaternions $\mathbb{H}$ form an algebra over both the real and the complex numbers. Note that it is not a commutative algebra since, for example $\mathbf{ij} = -\mathbf{ji}$.

Example 1.4. For $n > 2$, the $n \times n$-matrices $M_n(\mathbb{K})$, over a field $\mathbb{K}$), are a noncommutative algebra over $\mathbb{K}$.

Example 1.5. The octonions $\mathbb{O}$ form a non-associative algebra over both the real and the complex numbers.

1.2. Definitions and Examples of Lie Algebras.

Lie algebras are a very important family of nonassociative algebras. Before defining them we will recall how the motivating example of a Lie algebra can be constructed from an associative algebra.

Definition 1.6. Let A be an associative algebra over a field $\mathbb{K}$. Define the commutator
$$[-,-] : A \times A \to A, \qquad (x,y) \mapsto xy - yx.$$

Lemma 1.7. *This map satisfies*

1. $[-,-]$ *is bilinear,*
2. $[x,x] = 0,$ for all $x \in A,$
3. $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0,$ for all $x,y,z \in A.$

A very important point is that the bracket $[-,-]$ is *not* associative. The following definition (probably the most important in the course) axiomises the properties of the commutator bracket.

Definition 1.8. A *Lie algebra* over a field $\mathbb{K}$ is a pair $(L,[-,-])$, where L is a $\mathbb{K}$-vector space, and $[-,-]$ is a bilinear map

$$[-,-]: L \times L \to L$$

called the *Lie bracket*, satisfying

1. $[x,x] = 0$, for all $x \in L$, (anti-commutativity),
2. $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$, for all $x,y,z \in L$, (Jacobi identity).

Lemma 1.9. *For any Lie algebra L, it holds that*

$$[x,y] = -[y,x].$$

Conversely, for a vector space L over a field $\mathbb{K}$ of characteristic not equal to 2, a bilinear map $L \times L \to L$ satisfies $[x,y] = -[y,x]$ only if it is anti-commutative.

Proof. The anti-commutativity axiom of the definition of a Lie algebra implies that

$$0 = [x+y,x+y] = [x,x] + [x,y] + [y,x] + [y,y] = [x,y] + [y,x],$$

and so, we must have that $[x,y] = -[y,x]$.
Conversely, if $[x,y] = -[y,x]$, for all $x,y \in L$, then we have $[x,x] = -[x,x]$, or equivalently, we have $2[x,x] = 0$. Since $\mathrm{char}(\mathbb{K}) \neq 2$, this implies that $[x,x] = 0$. □

Definition 1.10. Let L be a Lie algebra. A *Lie subalgebra* of L is a subspace $M \subseteq L$ such that $[x,y] \in M$, for all $x,y \in M$.

Note that M equipped with the restricted bracket operation is itself a Lie algebra. We now consider a special case of the Lie algebra considered in Definition 1.6.

Example 1.11. Let V be a $\mathbb{K}$-vector space and $\mathrm{End}_{\mathbb{K}}(V)$ denote the space of linear endomorphisms of V (i.e. $\mathbb{K}$-linear maps from V to itself). This has an algebra structure, given by composition $\circ$, and so, we can consider the commutator bracket

$$[A,B] = A \circ B - B \circ A, \qquad \text{for } A,B \in \mathrm{End}_{\mathbb{K}}(V).$$

We call the pair $(\mathrm{End}_{\mathbb{K}}(V),[-,-])$ the *general linear Lie algebra* of V, and we usually denote it by $\mathfrak{gl}(V)$.
If V is finite-dimensional with dimension n, then choosing a basis of V gives a $\mathbb{K}$-linear isomorphism between $\mathrm{End}_{\mathbb{K}}(V)$ and $M_n(\mathbb{K})$, and the Lie bracket corresponds to the usual commutator of matrices. We usually denote the concrete Lie algebra $(M_n(\mathbb{K},[-,-])$ by $\mathfrak{gl}_n(\mathbb{K}))$.

Example 1.12. Consider the subspace

$$\mathfrak{sl}_n(\mathbb{K}) := \{A \in \mathfrak{gl}_n(\mathbb{K}) \mid \mathrm{tr}(A) = 0\} \subseteq \mathfrak{gl}_n(\mathbb{K}).$$

For any $A,B \in \mathfrak{gl}_n(\mathbb{K})$, we have that

$$\mathrm{tr}([A,B]) = \mathrm{tr}(A \circ B - B \circ A) = \mathrm{tr}(A \circ B) - \mathrm{tr}(B \circ A) = 0.$$

Therefore $\mathfrak{sl}_n(\mathbb{K})$ is closed under the Lie bracket, and hence it is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{K})$. We call it the *special linear algebra* of order n.

1.3. **Derivations and homomorphisms.** NOTE: The algebras in this subsection are not necessarily associative nor Lie algebras.

Definition 1.13. Let A be a $\mathbb{K}$-algebra. A linear map $\delta : A \to A$ is called a *derivation* if it satisfies the *Leibniz rule*

$$\delta(a.b) = \delta(a).b + a.\delta(b), \qquad \text{for all } a, b \in A.$$

The set of all derivations of A is denoted by $\mathrm{Der}(A)$.

Definition 1.14. An algebra homomorphism between Lie algebras is called a *Lie algebra homomorphism.* Explicitly, a linear map $f : L \to L$, for L a Lie algebra, is a Lie algebra homomorphism if

$$f([x, y]) = [f(x), f(y)], \qquad \text{for all } x, y \in L.$$

Lemma 1.15. *Let A be an algebra over a field $\mathbb{K}$. For $\delta, \delta' \in \mathrm{Der}(A)$, the commutator*

$$[\delta, \delta'] = \delta \circ \delta' - \delta' \circ \delta$$

is again a derivation. In particular, $\mathrm{Der}(A)$ *is a Lie subalgebra of* $\mathfrak{gl}(A)$.

Proof. Let $\delta, \delta' \in \mathrm{Der}(A)$. For any $x, y \in A$, it is easily checked that

$$[\delta, \delta'](xy) = [\delta, \delta'](x)y + x[\delta, \delta'](y),$$

and so, $[\delta, \delta'] \in \mathrm{Der}(A)$. Moreover, if $\delta, \delta' \in \mathrm{Der}(A)$ and $c \in \mathbb{K}$, then evidently

$$\delta + \delta' \in \mathrm{Der}(A), \qquad \text{and} \qquad c\delta \in \mathrm{Der}(A).$$

Thus we see that $\mathrm{Der}(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$. $\square$

Lemma 1.16. *Let L be a Lie algebra. Then, for any $x \in L$, the map*

$$\mathrm{ad}(x) =: \mathrm{ad}_x : L \to L, \qquad y \to [x, y]$$

is a derivation, called an inner derivation*. The map*

$$\mathrm{ad} : L \to \mathrm{Der}(L) \subseteq \mathfrak{gl}(L), \qquad x \mapsto \mathrm{ad}_x$$

is a homomorphism of Lie algebras, called the adjoint representation *of L.*

Proof. The Leibniz rule for ad_x can be written as

$$\mathrm{ad}_x([y, z]) = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)], \qquad \text{for all } y, z \in L.$$

Equivalently, this can be rewritten as

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

Using anti-commutativity, this can in turn be rewritten as

$$[x, [y, z]] = -[z, [x, y]] - [y, [z, x]]$$

which is clearly equivalent to the Jacobi identity. This implies that $\mathrm{ad}_x \in \mathrm{Der}(L)$. For any $\delta \in \mathrm{Der}(L)$, we have

$$[\delta, \mathrm{ad}_y](z) = \delta([y, z]) - \mathrm{ad}_y(\delta(z)) = [\delta(y), z] + [y, \delta(z)] - [y, \delta(z)] = [\delta(y), z] = \mathrm{ad}_{\delta(y)}(z).$$

Therefore $[\delta, \mathrm{ad}_y] = \mathrm{ad}_{\delta(y)}$. In particular, for the special case of $\delta = \mathrm{ad}_x$, we obtain

$$[\mathrm{ad}_x, \mathrm{ad}_y] = \mathrm{ad}_{\mathrm{ad}_x(y)} = \mathrm{ad}_{[x,y]}.$$

Thus we see that $\mathrm{ad} : L \to \mathrm{Der}(L)$ is a Lie algebra homomorphism. □

2. Ideals, Quotients, and Representations of Lie Algebra

In this section we will assume that $\mathrm{char}(\mathbb{K}) = 0$ and that all our Lie algebras and vector spaces are finite-dimensional.

2.1. Ideals.

Notation 2.1. Given a Lie algebra L and subspaces $A, B \subseteq L$, we denote

$$[A, B] \subseteq L := \mathrm{span}_{\mathbb{K}} \{[x, y] \mid x \in A,\, y \in B\}.$$

Definition 2.2. Let L be a Lie algebra.

1. A subspace $I \subseteq L$ is called an *ideal* of L if for any $x \in L$ and $y \in I$, we have $[x, y] \in I$, or equivalently $[L, I] \subseteq I$.
2. The *center* of L is
$$Z(L) = \{x \in L \mid [x, y] = 0, \text{ for all } y \in L\} = \{x \in L \mid [x, L] = 0\}.$$
3. The *derived Lie algebra* of L is $[L, L]$.
4. A Lie algebra L is called *abelian*, or *commutative*, if $[x, y] = 0$, for all $x, y \in L$, or equivalently if $[L, L] = 0$.

Lemma 2.3.

1. *Every ideal $I \subseteq L$ is also a Lie subalgebra of L.*
2. *$[L, I] \subseteq I$ implies also $[I, L] \subseteq I$.*
3. *The center $Z(L)$ is an ideal of L.*
4. *The* derived algebra *$[L, L]$ is an ideal of L.*

Proof. This is left as a simple exercise. □

Note: It is important to note that point 2 of the lemma means that **there is no difference between right and left ideals**.

Example 2.4. If $f : L \to L$ is a Lie algebra homomorphism, then $\ker(f)$ is an ideal of L . Indeed, given any $x \in L$ and $y \in \ker(f)$, we have

$$f([x, y]) = [f(x), f(y)] = [f(x), 0] = 0.$$

Thus $[x, y] \in \ker(f)$.

Example 2.5. If L is abelian then any subspace $I \subseteq L$ is an ideal of L.

Definition 2.6. Given an ideal $I \subseteq L$, the quotient vector space L/I can be equipped with a Lie algebra structure as follows: the bracket on L/I is defined by

$$[x + I, y + I] = [x, y] + I, \qquad \text{for all } x, y \in L.$$

This Lie algebra is called the *quotient Lie algebra* of L by I.

For any such quotient, the map

$$\mathrm{proj} : L \to L/I, \qquad\qquad x \to [x + I]$$

is clearly a Lie algebra homomorphism.

Note: The quotient as a vector space is just the usual quotient of vector spaces. You should convince yourself that the Lie bracket is well-defined as a bilinear map.

Example 2.7. Let

$$L = \mathfrak{b}_n(\mathbb{K}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in \mathbb{K} \right\}$$

be the Lie algebra of upper-triangular matrices and

$$I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{K} \right\}.$$

Then I is an ideal of L and L/I is isomorphic to the abelian algebra $\mathbb{K}$ of diagonal matrices.

Lemma 2.8. *Let L be a Lie algebra, and let I and J be two ideals of L, then $[I, J]$ is also an ideal of L.*

Proof. This follows directly from the Jacobi identity. □

2.2. **Representations of Lie Algebras.** In previous sections we have introduced the definition of a Lie algebra as a generalisation of the properties of the commutator bracket of the space of linear operators on a vector space. A very important part of Lie theory is to find concrete realisations of Lie algebras as linear operators on some vector space, this allows us to apply the tools of linear algebra to study an abstract Lie algebra. This process is formalised in the notion of a *representation*, which we now present.

Definition 2.9. Let L be a Lie algebra, over a field $\mathbb{K}$.

1. A *representation of* L is a pair (V, ρ), where V is a $\mathbb{K}$-vector space and $\rho : L \to \mathfrak{gl}(V)$ is a Lie algebra homomorphism.
2. An *L-module* (or *module over* L) is a vector space V with a bilinear map
$$L \times V \to V, \qquad\qquad (x, v) \mapsto xv,$$
such that $[x, y]v = xyv - yxv$, for all $x, y \in L, v \in V$.
3. For a representation or a module we say that L *acts* on V.

Lemma 2.10. *There is a bijective correspondence between representations of L and L-modules. Given a representation (V, ρ), we can equip V with an L-module structure by setting*

$$(x, v) \mapsto xv := \rho(x)v, \qquad\qquad \text{for } x \in L, v \in V.$$

Proof. This map satisfies the axiom of an L-module, as is clear from

$$[x, y]v = \rho([x, y])v = [\rho(x), \rho(y)]v = (\rho(x)\rho(y) - \rho(y)\rho(x))v = xyv - yxv.$$

The fact that this gives a bijective correspondence we leave as an exercise. □

Example 2.11. Any vector space V, so in particular $\mathbb{K}$, can be equipped with the *trivial action* of L defined by

$$xv = 0, \qquad \text{for all } x \in L,\ v \in V.$$

Example 2.12. For any vector space V, the general linear algebra $\mathfrak{gl}(V)$ acts on V in an obvious way. This representation is called the *standard representation*, or *vector representation*, of $\mathfrak{gl}(V)$.

Example 2.13. We have seen that the map

$$\mathrm{ad} : L \to \mathfrak{gl}(L), \qquad x \mapsto \mathrm{ad}_x,$$

is a Lie algebra homomorphism. We call it the *adjoint representation* of L. Note that

$$\ker(\mathrm{ad}) = \{x \in L | \mathrm{ad}_x = 0\} = \{x \in L | [x, L] = 0\} = Z(L)$$

is the center of L.

This is in fact a special example of the following general result.

Lemma 2.14. *For (V, ρ) a representation of a Lie algebra L, the kernel*

$$\ker(\rho) = \{x \in L \,|\, \rho(x) = 0\}$$

is an ideal of L.

Proof. For $x \in \ker(\rho)$, and $y \in L$, we have that

$$\rho([x, y]) = [\rho(x), \rho(y)](v) = 0.$$

Thus we see that $[x, y] \in \ker(\rho)$, which is to say, $\ker(\rho)$ is an ideal of L. □

Definition 2.15. Let L be a Lie algebra.

1. A linear map $f : V \to W$ between L-representations is called a *module map* if
   $$f(xv) = xf(v), \qquad \text{for all } x \in L, v \in V.$$
   It is called a *isomorphism* of modules if it is an isomorphism of vector spaces, that is, if it is bijective.
2. Given an L-module V , a subspace $W \subseteq V$ is called a *submodule* of V if $xv \in W$, for all $x \in L$, $v \in W$.
3. Given an L-representation V, an element $v \in V$ is called L-invariant if $xv = 0$, for all $x \in L$. The vector space of all L-invariant elements of V , which is a submodule of V, is denoted by V^L.

We can use submodules to construct new representations, as we see in the following lemma.

Lemma 2.16. *If V is an L-module and $W \subseteq V$ is a subrepresentation, then the quotient space V/W can be equipped with the structure of an L-representation*

$$x(v + W) = xv + W, \qquad \textit{for } x \in L, v \in V.$$

Proof. This is left as an easy exercise. □

3. Nilpotent and Solvable Lie Algebras

In this section we introduce nilpotent, and more generally solvable, Lie algebras. These Lie algebras are, in some sense, at the opposite end of the spectrum from semisimple Lie algebras that we will meet in the next section.

3.1. Basics of Nilpotent Lie Algebras.

Recalling the derived subalgebra of a Lie algebra considered in Lemma 2.3, we see that the process can be iterated to give a series of ideals in L. This observation gives us the definition of nilpotent Lie algebras, which comprise an extremely important class of Lie algebras.

Definition 3.1. Let L be a finite-dimensional Lie algebra.

1. Define the *central series* of ideals in L is given by
$$L^0 := L, \qquad L^n := [L, L^{n-1}], \qquad \text{for } n \in \mathbb{Z}_{\geq 1}.$$
2. L is called nilpotent if $L^m = 0$, for some $m \in \mathbb{Z}_{\geq 0}$.

Note that by Lemma 2.8 the terms in the central series are ideals of L.

Example 3.2. The simplest example of a nilpotent Lie algebra is an abelian Lie algebra.

Lemma 3.3. *For $n \geq 1$, the following conditions are equivalent*

1. $L^n = 0$.
2. $[x_1, [x_2, ..., [x_n, x_{n+1}]]] = 0$, *for all* $x_1, \ldots, x_{n+1} \in L$.
3. $\mathrm{ad}_{x_1} \circ \cdots \circ \mathrm{ad}_{x_n} = 0$, *for all* $x_1, \cdots, x_n \in L$. *In particular, if L is nilpotent then* ad_x *is a nilpotent operator, which is to say,* $\mathrm{ad}_x^n = 0$, *for all* $x \in L$.

Example 3.4. Let
$$L = \{(a_{ij}) \in \mathfrak{gl}_n(\mathbb{K}) \mid a_{ij} = 0 \text{ for } i \geq j\} \subseteq \mathfrak{gl}_n(\mathbb{K})$$
be the Lie algebra of upper-triangular matrices. Let us show that L is nilpotent. The first term $L^0 = L$ is spanned by matrices E_{ij} with $j > i$ (where E_{ij} has 1 on the intersection of i-th row and j-th column and zero otherwise). For any $i < j, k < l$, the second term in the derived series $L^1 = [L, L]$ is generated by
$$[E_{ij}, E_{kl}] = E_{ij}E_{kl} - E_{kl}E_{ij} = \delta_{jk}E_{il} - \delta_{li}E_{kj}.$$
Now we can assume, without loss of generality, that $i \neq l$ (convince yourself that this is true). Thus we see that $[L, L]$ is spanned by elements of the form E_{il}, such that $l - i \geq 2$. This means that E_{il} lives at least two vertical positions above the diagonal. Continuing this process we obtain that L^k is spanned by the matrices E_{ij}, with $j - i \geq k + 1$. This implies that $L^{n-1} = 0$ and hence that L is nilpotent.

3.2. Engel's Theorems.

We begin with two simple lemmas that will be used in the proof of the main theorem of this section, that is, Engel's theorem, one of the basic results about nilpotent Lie algebras.

Lemma 3.5. *Let L be a Lie subalgebra of* $\mathrm{End}_{\mathbb{K}}(V)$ *consisting of nilpotent operators. Then L acts nilpotently on itself, with respect to the adjoint representation.*

Proof. For any $x \in L$, consider the operators

$$L_x : \mathfrak{gl}(V) \to \mathfrak{gl}(V), \qquad y \mapsto x \circ y$$
$$R_x : \mathfrak{gl}(V) \to \mathfrak{gl}(V), \qquad y \mapsto y \circ x.$$

Since each $x \in L$ acts nilpotently on V by assumption, there exists an $n \in \mathbb{Z}_{>0}$ such that $x^n = 0$. Therefore we must have that $L_x^n = R_x^n = 0$. Moreover, since

$$L_x \circ R_x(y) = L_x(y \circ x) = x \circ y \circ x = R_x(x \circ y) = R_x \circ L_x(y),$$

we see that the two operators L_x and R_x commute. Note next that

$$\mathrm{ad}_x(y) = xy - yx = L_x(y) - R_x(y).$$

By the above properties of L_x and R_x, we have that

$$\mathrm{ad}_x^{2n}(y) = \sum_{a=0}^{2n} (-1)^a \binom{2n}{a} L_x^{2n-a} \circ R_x^a(y) = 0, \qquad \text{for all } y \in L.$$

Thus L acts nilpotently on itself. □

Lemma 3.6. *Every Lie algebra of dimension greater than* 2 *admits a proper Lie subalgebra.*

Proof. Take any $x \in L$. Since $[x, x] = 0$ the subspace $\mathbb{K}x$ is a one-dimensional Lie subalgebra of L. Moreover, since $\dim(L) \geq 2$, it is a proper Lie subalgebra. □

Definition 3.7. For any Lie algebra L, and any Lie subalgebra $S \subseteq L$, the subspace

$$N(S) = \{x \in L \,|\, [x, S] \subseteq S\} \subseteq L$$

is called the *normalizer* of I.

Lemma 3.8. *The normailser of a Lie subalgebra $S \subseteq L$ is a Lie subalgebra of L. Moreover, S is an ideal of $N(S)$.*

Proof. This is left as an exercise. □

Lemma 3.9. *Let L be a Lie algebra and (V, ρ) a non-zero finite-dimensional representation of L such that $\rho(x) \in \mathrm{End}_{\mathbb{K}}(V)$ is nilpotent, for all $x \in L$. Then there exists a non-zero $v \in V$ such that $\rho(x)v = 0$, for all $x \in L$, that is, a non-zero L-invariant element.*

Proof. To ease notation, we will denote the Lie subalgebra $\rho(L) \subseteq \mathrm{End}_{\mathbb{K}}(V)$ by K. Moreover, we note that since any $x \in K$ is nilpotent by assumption, there must exist a $v \in V$ such that $xv = 0$. (Thus every element of K has a non-trivial kernel, the thing we need to prove is that the all the elements x have a *common* v in their kernels.)

We will prove the result by induction on the dimension of the Lie algebra K. Assume first that $\dim(K) = 1$, or equivalently that for any $x \in K$, we have $K = \mathbb{C}x$. Then for any $v \in \ker(x)$, we have $(\lambda x)v = 0$, which is to say $Kv = 0$.

Let us now assume that the result holds true for all K of dimension less than or equal to n, for some $n \geq 2$. Let $I \subseteq K$ be a maximal proper subalgebra (since $\dim(K) \geq 2$, such an I exists by Lemma 3.6). Let us show that I is an ideal of K: Take the normaliser

$$N(I) = \{x \in K \,|\, [x, I] \subseteq I\}.$$

Just as for any normaliser, we have $I \subseteq N \subseteq K$. If we could establish that $N = K$, then Lemma 3.8 would imply that I was an ideal of K. By maximality of I, it is enough to show that I is properly contained in N. To do so we use the inductive hypothesis: It follows from Lemma 3.5 that K acts on itself nilpotently, and so, the Lie subalgebra I must act on K/I nilpotently (note that this action is well-defined since I is a Lie subalgebra of L). Since $\dim(I) < \dim(K)$, our inductive hypothesis implies that there exists a non-zero element $x + I \in K/I$ such that

$$\mathrm{ad}_m(x + I) = 0, \qquad\qquad \text{for all } m \in I.$$

This of course then implies that

$$[x, I] \subseteq I \quad \Rightarrow \quad x \in N(I).$$

Therefore, since $x \notin I$ (otherwise we would have $x + I = 0$) we must have that I is properly contained in N. Thus we see that I is an ideal of K.

Consider next the vector subspace

$$W = V^I = \{v \in V \mid Iv = 0\} \subseteq V.$$

Since $\dim(I) < \dim(K)$, our inductive hypothesis implies that W contains a non-zero element. Moreover, W is a K-submodule of V, as we see from

$$mxv = xmv - [x, m]v = 0, \qquad\qquad \text{for all } m \in I,\ x \in K,\ v \in W.$$

This means that W is a K-module satisfying

$$IW = 0.$$

Thus we have a well-defined action of the quotient Lie algebra K/I on W:

$$K/I \times W \to W, \qquad\qquad (k + I, w) \mapsto kw.$$

Since $\dim(K/I) < \dim(K)$, our inductive hypothesis implies that there exists a $v \in W$ such that $(K/I)v = 0$. Thus we see that $Kv = 0$, giving us the required vector $v \in W \subseteq V$. $\square$

Theorem 3.10. *Let L be a Lie algebra and (V, ρ) be a non-zero finite-dimensional representation of L such that $\rho(x) \in \mathrm{End}(V)$ is nilpotent, for all $x \in L$.*

1. *There exists a basis of V such that each $\rho(x)$ is strictly upper-triangular with respect to this basis.*
2. *There exists a filtration by subrepresentations*

$$0 =: V_0 \subset V_1 \subset \cdots \subset V_n := V$$

such that $LV_k \subset V_{k-1}$, or equivalently such that L acts trivially on the quotient module V_k/V_{k-1}, for $1 \le k \le n$. Moreover, $\dim(V_k/V_{k-1}) = 1$.

Proof.
1. Recall from the theorem above, that we have a v in the common kernel of the elements of $\rho(L)$. Noting that $\mathbb{K}v \subset V$ is a subrepresentation of V, we can apply our inductive assumption to the quotient representation $V' := V/\mathbb{K}v$. This gives a basis

$$v_2, \ldots, v_n$$

of V', with respect to which the elements of L act as strictly upper-triangular matrices. Consider now a collection of elements $v_2, v_3, \ldots, v_n \in V$ satisfying $\pi(v_i) = v_i'$, where $\pi : V \to V/\mathbb{K}v_1$ where π is the canonical projection. The set

$$\{v_1 := v, v_2, \ldots, v_n\}$$

forms a basis of V. Now with respect to this basis, $\rho(x)$ is strictly upper triangular (you should think about this).

2. We define $V_k := \mathrm{span}_{\mathbb{K}}\{v_1, \ldots, v_k\}$, for $1 \leq k \leq n$. As the matrix of $\rho(x)$ is strictly upper-triangular, it maps v_k to a linear combination of $v_1, \ldots, v_{k1}$. This implies that $\rho(x)V_k \subset V_{k-1}$. Moreover, we see that $\dim(V_k/V_{k-1}) = 1$. □

Theorem 3.11 (Engel)**.** *A finite-dimensional Lie algebra L is nilpotent if and only if $\mathrm{ad}_x \in \mathfrak{gl}(L)$ is a nilpotent operator for all $x \in L$.*

Proof. If $L^n = 0$ then $\mathrm{ad}_x^n = 0$, for all $x \in L$. Conversely, assume that ad_x is nilpotent for every $x \in L$. Applying Corollary 3.10 to the adjoint representation $\mathrm{ad} : L \to \mathfrak{gl}(L)$, we can find a filtration by subspaces

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = L$$

such that $\mathrm{ad}_x(V_k) \subseteq V_{k-1}$, for all $x \in L$ and $k \geq 1$. In other words, $LV_k \subseteq V_{k-1}$, meaning that the n-fold Lie bracket satisfies $[L, [L, [\cdots] \cdots] = 0$. □

3.3. **Basics of Solvable Lie Algebras.** In this subsection we introduce an important generalisation of the definition of a nilpotent Lie algebra.

Definition 3.12. Let L be a Lie algebra.

1. The *derived series* of ideals in L is given by
$$L^{(0)} = L, \qquad L^{(n)} = [L^{(n-1)}, L^{(n-1)}], \qquad \text{for } n \geq 1.$$
2. L is called *solvable* if $L^{(n)} = 0$, for some $n \in \mathbb{Z}_{n\geq 1}$.

Note: We note that solvable implies nilpotent, but the other implication is not true in general. In other words, solvable is a weaker requirement than nilpotent.

Lemma 3.13. *Let L be a Lie algebra.*

1. *If $I \subseteq L$ is an ideal and L is solvable then $I, L/I$ are solvable.*
2. *If $I \subseteq L$ is an ideal and $I, L/I$ are solvable then L is solvable.*
3. *If $I, J \subseteq L$ are solvable ideals then $I + J$ is solvable.*

Proof.
1. Let $L^{(n)} = 0$. Then $I^{(n)} \subseteq L^{(n)} = 0$ and I is solvable. Let $\pi : L \to L/I$ be the canonical projection, and let us prove by induction that $\pi(L^{(k)}) = (L/I)^{(k)}$: The equality clearly holds for $n = 0$. Assuming that it holds for $n = k$, we see that

$$\pi(L^{(k+1)}) = \pi\left([L^{(k)}, L^{(k)}]\right) = [\pi(L^{(k)}), \pi(L^{(k)})] = [(L/I)^{(k)}, (L/I)^{(k)}] = (L/I)^{(k+1)}.$$

In particular $(L/I)^{(n)} = \pi(L^{(n)}) = 0$ and L/I is solvable.

2. Assume that $I^{(m)} = 0$ and $(L/I)^{(n)} = 0$. Then $\pi(L^{(n)}) = (L/I)^{(n)} = 0$ in L/I and therefore $(L)^{(n)} \subseteq I$. This implies that $L^{(m+n)} \subseteq I^{(m)} = 0$ and hence that L is solvable.

3. There is a natural isomorphism

$$(I + J)/I \simeq J/(I \cap J).$$

If J is solvable then also the quotient $J/(I \cap J)$ is solvable by (1). Therefore $(I + J)/I$ is solvable and if I is solvable then $I + J$ is solvable by (2). $\square$

3.4. **Lie's Theorem.** In this subsection we prove Lie's theorem. Note that we need to assume that our field $\mathbb{K}$ has characteristic zero. For non-zero characteristic Lie's theorem fails in general.

Lemma 3.14. *Let L be a Lie algebra over a field $\mathbb{K}$ of characteristic zero, $I \subseteq L$ an ideal, and V a finite-dimensional representation of L. For any linear map $\lambda : I \to \mathbb{K}$, consider the space*

$$V_\lambda = \{v \in V \mid xv = \lambda(x)v, \ \textit{for all } x \in I\}.$$

It holds that

1. $[L, I]V_\lambda = 0$,
2. *V_λ is an L-submodule of V.*

Proof. If $V_\lambda = 0$ then both claims holds trivially, so we will assume that V_λ is non-zero.

1. Choose some $x \in L$, and note that since I is an ideal, $[x, y] \in I$, for all $y \in I$, and so $[x, y]$ acts on V_λ as $\lambda([x, y])$. Thus we want to show that $\lambda([x, y]) = 0$, for all $x \in L$, and $y \in I$. Let $0 \neq w \in V_\lambda$ and denote $w_k := x^{k-1}w$, for $k \geq 1$. Let us show that

$$yw_k = \lambda(y)w_k + \sum_{i=1}^{k-1} a_i w_i, \qquad \text{for any } y \in I, \tag{1}$$

by induction, where $a_i \in \mathbb{K}$ are some constants which depend on y and k. Note first that, for $k = 1$, we have $yw = \lambda(y)w$. Assuming that the identity holds for k, we see

$$yw_{k+1} = yx^k w = \Big(xy - [x, y]\Big)x^{k-1}w = \Big(xy - [x, y]\Big)w_k = \ xyw_k - [x, y]w_k.$$

By the inductive hypothesis, it holds that

$$xyw_k = x\left(\lambda(y)w_k + \sum_{i=1}^{k-1} a_i w_i\right) = \lambda(y)w_{k+1} + \sum_{i=1}^{k-1} a_i w_{i+1},$$

and moreover that

$$-[x, y]w_k = -\sum_{i=1}^{k} a'_i w_i.$$

(We note that $a'_k = \lambda([x, y])$, although we will not need to use this fact). Collecting these terms together we get that

$$yw_{k+1} = \lambda(y)w_{k+1} + \sum_{i=1}^{k} b_i w_i.$$

Thus by induction the identity holds for all k. Let us now consider U, the vector space spanned by the elements w_k, for $k \in \mathbb{Z}_{>0}$. Note that since V is finite-dimensional, U is necessarily also finite-dimensional, meaning in particular that there exists a maximal n such that the set

$$B := \{w_1, w_2, \dots, w_n\}$$

is linearly independent. Clearly, by the construction of U it holds that $xU \subseteq U$.

Equation (1) now implies that, with respect to the basis B, every $y \in I$ acts on U as upper triangular matrices with the diagonal entries given by $\lambda(y)$. From this we see

$$\mathrm{tr}(y) = n\lambda(y), \qquad \text{for all } y \in I.$$

Since I is an ideal, the previous identity can be applied to $[x, y] \in I$, which is to say

$$0 = \mathrm{tr}([x, y]) = n\lambda([x, y]),$$

where we have used the fact that the trace of the commutator of two operators is always zero. Since our field $\mathbb{K}$ is by assumption of characteristic zero, it must hold that $\lambda([x, y]) = 0$. Thus $[I, L]V_\lambda = 0$ as claimed.

2. For any $x \in L$, $y \in I$, we now know that $[x, y]v = 0$, for all $v \in V_\lambda$. This means

$$yxv = xyv - [x, y]v = xyv = \lambda(y)xv,$$

proving that $xv \in V_\lambda$. Thus we see that V_λ is an L-submodule of V. □

Theorem 3.15 (Lie's Theorem)**.** *Let L be a solvable Lie algebra over an algebraically closed field of characteristic 0, and let (ρ, V) be an n-dimensional non-zero representation of L. Then*

1. *there exists a common eigenvector $0 \neq v \in V$ of L, that is, an eigenvalue of $\rho(x)$, for all $x \in L$, which is to say, $\mathbb{K}v$ is a subrepresentation of L,*
2. *there exists a basis of V such that $\rho(x)$ is upper-triangular in this basis, for all $x \in L$,*
3. *there exists a chain of subrepresentations*

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V,$$

such that $\dim(V_k/V_{k-1}) = 1$, for $1 \leq k \leq n$.

Proof. 1. Since L is solvable, we have $[L, L] \neq L$. Indeed if

$$L^{(1)} = [L, L] = L,$$

then we would have

$$L^{(2)} = [L^{(1)}, L^{(1)}] = [L, L] = L,$$

and more generally $L^{(k)} = L$, contradicting our assumption that L is solvable.

This means that we can choose a codimension 1 subspace I satisfying

$$[L, L] \subseteq I \subseteq L.$$

Note that since $[L, L] \subseteq I$ we must have that I is an ideal.

Let us now follow an inductive argument on the dimension of L. Note first that 1. is trivially true for $L = 0$. Assume next that 1. holds for all Lie algebras of dimension less than that of L. This means, in particular, that there exists a common eigenvector $v \in V$, for all $y \in I$. Or equivalently, there exists a linear function $\lambda : I \to \mathbb{K}$ such that

$$yv = \lambda(y)v, \qquad \text{for all } y \in I.$$

From this we see that the space

$$W = \{v \in V \mid yv = \lambda(y)v, \quad \text{for all } y \in I\}$$

is non-zero. Note that by the previous lemma, W is necessarily an L-submodule of V. Given an $x \in L$, such that $x \notin I$, the map

$$x : W \to W, \qquad w \mapsto xw$$

has a non-zero eigenvector $v \in W$ (since we are assuming that $\mathbb{K}$ is algebraically closed). In fact, the definition of W implies that v is a common eigenvector of $\mathbb{K}x + I = L$.

($1 \implies 2$) We will use an inductive argument on the dimension of V. Clearly, the result holds when $\dim(V) = 1$. Let us now assume that it holds for all representations of dimension less than n. Denoting $v_1 := v$ (the common eigenvector of L), we can consider the $(n-1)$-dimensional quotient representation

$$V' := V/\mathbb{K}v_1.$$

By the inductive hypothesis V' admits a basis $(v_2', \dots, v_n')$ with respect to which x is an upper triangular matrix, for all $x \in L$. Lifting this basis to some vectors $v_2, \dots, v_n$ in V, we see that $(v_1, v_2, \dots, v_n)$ is a basis of V with respect to which $\rho(x)$ is upper triangular.

($2 \implies 3$) Choose a basis as above and define

$$V_k := \operatorname{span}_{\mathbb{K}}\{v_1, \dots, v_k\}.$$

The fact that each matrix $\rho(x)$ is upper triangular implies that each V_k is a subrepresentation of V. Moreover, by construction the dimension condition is satisfied. □

Corollary 3.16. *Let L be a Lie algebra over an algebraically closed field of characteristic zero. Then L is solvable if and only if $[L, L]$ is nilpotent.*

Proof. Let us first assume that L is solvable. Consider the adjoint representation $\mathrm{ad} : L \to \mathfrak{gl}(L)$. It follows from Lie's theorem that we can find a basis of L such that all operators ad_x are upper triangular. Recall that the commutator of any two upper triangular matrices is strictly upper triangular (you should prove this if you didn't know it already). From this we see that

$$\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]$$

is a strictly upper triangular matrix. Hence, for every $z \in [L, L]$ the operator ad_z is strictly upper triangular, and so a nilpotent operator. It now follows from Engel's second theorem that $[L, L]$ is a nilpotent Lie algebra.

Conversely, if $[L, L]$ is a nilpotent Lie algebra then it is solvable. Hence

$$L^{(n+1)} = ([L, L])^{(n)} = 0,$$

for some $n \geq 1$, hence L is solvable. □

4. Simple, Semisimple and Reductive Lie Algebras

In this section we consider a new family of Lie algebras, namely simple, semisimple, and reductive Lie algebras. These Lie algebras are characterised by having far fewer ideals than solvable or nilpotent Lie algebras, and in a sense explained below, form a complementary class of Lie algebras, at the opposite extreme to solvable Lie algebras.

Throughout this section, all Lie algebras will be **assumed to be finite-dimensional**.

4.1. **Basic Definitions.** We start with the definition of the radical of a finite-dimensional Lie algebra L, the maximal solvable ideal of L.

Definition 4.1. For a finite-dimensional Lie algebra L, it follows from Lemma 3.13 that a maximal solvable ideal of L is given by

$$\operatorname{rad}(L) := \sum_{I \in \operatorname{Solv}(L)} I,$$

where $\operatorname{Solv}(L)$ denotes the set of solvable ideals of L. We call $\operatorname{rad}(L)$ the *radical* of L.

As an immediate consequence of the definition we get the following lemma.

Lemma 4.2. *Let L be a Lie algebra.*

1. *Since the center $Z(L)$ is abelian, it is solvable and hence $Z(L) \subseteq \operatorname{rad}(L)$.*
2. *If $\operatorname{rad}(L) = L$, then L must be a solvable Lie algebra.*

Definition 4.3. Let L be a Lie algebra

1. L is **simple** if it is not abelian and it does not contain ideals except 0 and L,
2. L is **semisimple** if it does not contain any non-zero abelian ideals,
3. L is called **reductive** if $Z(L) = \operatorname{rad}(L)$.

Remark 4.4. Let L be an abelian Lie algebra L of dimension strictly greater than 1, then for any $x \in L$, the proper subspace $\mathbb{K}x$ is an ideal of L. Hence L cannot be simple. Thus we see that the requirement that a simple Lie algebra be non-abelian is equivalent to requiring that L is not equal to the unique one-dimensional (necessarily abelian) Lie algebra $\mathfrak{gl}_1$.

Lemma 4.5. *If a Lie algebra L contains a non-zero solvable ideal I, then L contains an abelian idea J such that $J \subseteq I \subseteq L$. In particular, every solvable Lie algebra contains a non-trivial abelian ideal.*

Proof. Let $I \subseteq L$ be a non-zero solvable ideal. Then there exists an $n \geq 1$ such that $I^{(n)} = 0$ and $I^{(n-1)} \neq 0$. We have

$$[I^{(n-1)}, I^{(n-1)}] = I^{(n)} = 0,$$

hence $I^{(n-1)}$ is a non-zero abelian ideal of L. □

Lemma 4.6. *Let L be a Lie algebra. The following conditions are equivalent:*

1. *L is semisimple,*
2. *L does not contain any non-zero solvable ideals,*

3. $\mathrm{rad}(L) = 0$.

Proof.
($1 \implies 2$): By the above lemma, if L contained a non-zero solvable ideal, then it would contain a non-zero abelian ideal, contradicting our assumption that L is semisimple.

($2 \implies 1$): Since any abelian ideal is automatically solvable, if L does not contain any non-zero solvable ideals, then it cannot contain any non-trivial abelian ideals.

($3 \Leftrightarrow 1$) This is now clear from the equivalence of 1 and 2. □

Lemma 4.7. *It holds that* ***simple*** $\implies$ ***semisimple*** $\implies$ ***reductive****.*

Proof. The first implication is clear. The second implication follows from the fact that $Z(L)$ is contained in the radical, and so, if we have zero radical, then clearly the center and the radical coincide. □

Corollary 4.8. *If L is abelian, then it is reductive. Hence abelian Lie algebras are both solvable and reductive.*

Proof. If L is abelian then $L = Z(L) = \mathrm{rad}(L)$, which is to say, L is reductive. □

Let us now summarise these facts in the following table:

Ideal Description $0 \subseteq Z(L) \subseteq \mathrm{rad}(L) \subseteq L$	**Implication**	**Lie algebra Property**
$Z(L) = \mathrm{rad}(L) = L$	$\Leftrightarrow$	abelian
$\mathrm{rad}(L) = L$	$\Leftrightarrow$	solvable
$0 = Z(L) = \mathrm{rad}(L)$	$\Leftarrow$	simple
$0 = Z(L) = \mathrm{rad}(L)$	$\Leftrightarrow$	semisimple
$Z(L) = \mathrm{rad}(L)$	$\Leftrightarrow$	reductive

Remark 4.9. Thus we see that the radical gives us a way of thinking about the semisimple and solvable Lie algebras as being at either end of the *spectrum* of Lie algebras. At one extreme we have that the radical is trivial, which is to say L is semisimple. At the other end we have that the radical everything, that is $\mathrm{rad}(L) = L$, that is, L is solvable. In the middle we have all those Lie algebra for which the radical is a proper ideal of L.

4.2. **Reductivity Criterion.** In this subsection we produce criteria for demonstrating that Lie algebras are reductive or semisimple.

Corresponding to the idea of simplicity for a Lie algebra, we have the notion of simplicity for a Lie algebra representation.

Definition 4.10. A representation (V, ρ) of a Lie algebra L is said to be **simple**, or **irreducible**, if the only L-submodules of V ore 0 and V itself.

A representation (V, ρ) of a Lie algebra L is said to be **faithful** if $\rho : L \to \mathrm{End}(V)$ is an injective map.

Lemma 4.11. *Let (V, ρ) be an irreducible L-representation, then there exists a linear map $\lambda : \mathrm{rad}(L) \to \mathbb{K}$ such that $xv = \lambda(x)v$, for all $x \in \mathrm{rad}(L)$.*

Proof. Since the radical is by construction solvable, Lie's theorem tells us that there exists a $v_0 \in V$ which is a common eigenvector of all the elements of $\mathrm{rad}(L)$. Equivalently, this means that we have a linear map $\lambda : \mathrm{rad}(L) \to \mathrm{K}$ such that $xv_0 = \lambda(x)v_0$, for all $x \in \mathrm{rad}(L)$. Let us denote

$$W_\lambda := \{v \in V \,|\, xv = \lambda(x)v, \quad \text{for all} \quad x \in L\}.$$

It follows from Lemma 3.14 that W_λ is necessarily an L-submodule of V. However, since V is irreducible, and $W_\lambda \neq 0$, we must have that $V = W_\lambda$. □

Theorem 4.12. *Let L be a Lie algebra, over a field of characteristic zero, admitting a faithful irreducible representation (V, ρ).*

1. *Then L is reductive and $\dim(Z(L)) \leq 1$.*
2. *If $\rho(L) \subseteq \mathfrak{sl}(V)$, then L is semisimple.*

Proof. Since the representation is faithful, we can identify L with its image in $\mathfrak{gl}(V)$. By the above lemma, there exists a $\lambda : \mathrm{rad}(L) \to \mathbb{K}$ every $x \in \mathrm{rad}(L)$ acts on V as $\lambda(x)\mathrm{id}_V$. Hence

$$\mathrm{rad}(L) \subseteq \mathbb{K}\mathrm{id}_V \subseteq Z(L),$$

which since the opposite inclusion is clear, means that L is reductive.

Moreover, since $\mathrm{rad}(L) = Z(L)$ is contained in a 1-dimensional space, it has at most dimension 1.

Assume now that $L \subseteq \mathfrak{sl}(V)$. If $\mathrm{rad}(L) \neq 0$, then it would be equal to $\mathbb{K}\mathrm{id}_V$. But this would imply

$$\mathrm{id}_V \in \mathrm{rad}(L) \subseteq \mathfrak{sl}(V),$$

which is of course a contradiction. Thus we must conclude that $\mathrm{rad}(L) = 0$ and hence that L is semisimple. □

Example 4.13. The Lie algebras $\mathfrak{gl}_n(\mathbb{K})$ and $\mathfrak{u}_n$ are reductive, and the Lie algebras $\mathfrak{sl}_n(\mathbb{K})$ and $\mathfrak{su}_n$ are semisimple.

Now we move onto a different approach to establishing reductivity for a Lie algebra. This time we will use bilinear forms to verify the property.

Definition 4.14. Let $\sigma : L \times L \to \mathbb{K}$ be a bilinear form. Then

1. σ is called **symmetric** if $\sigma(x, y) = \sigma(y, x)$, for all $x, y \in L$,
2. σ is called **invariant** if $\sigma([x, y], z) = \sigma(x, [y, z])$, for all $x, y \in L$,
3. if σ is symmetric, the we say that it is **non-degenerate** if for any $x \in L$, we have $\sigma(x, L) = 0$ if and only if $x = 0$.

Lemma 4.15. *Let $\sigma : L \times L \to \mathbb{K}$ be a bilinear L-invariant form. Then for every ideal $I \subseteq L$, an ideal is given by*

$$I^{\perp} := \{x \in L \,|\, \sigma(x, y) = 0, \;\; \text{for all } y \in I\}.$$

Proof. Exercise. □

Lemma 4.16. *Let (V, ρ) be an L-representation. Then the bilinear form*

$$\sigma_V : L \times L \to \mathbb{K}, \qquad (x, y) \mapsto \mathrm{tr}(\rho(x) \circ \rho(y))$$

is symmetric and L-invariant.

Proof. Exercise. □

Definition 4.17. We call the symmetric L-invariant bilinear form associated to the adjoint representation, which is to say the bilinear form

$$\kappa = \kappa_L : L \times L \to \mathbb{K}, \qquad (x, y) \mapsto \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y),$$

the **Killing form** of L.

Exercise 4.18. For (V, ρ) an L-representation, let U be a L-submodule. We can consider the sub-representation (U, ρ_U), and the quotient representation $(V/U, \rho_{V/U})$. Show that

$$\sigma_V = \sigma_U + \sigma_{V/U}.$$

(Hint: Take the linear isomorphism

$$V \simeq U \oplus U/V$$

and show that tr_V is given on the direct sum by $\mathrm{tr}_U \oplus \mathrm{tr}_{V/U}$.)

Theorem 4.19. *Let L be a Lie algebra over a field $\mathbb{K}$ of characteristic zero and let (V, ρ) be a finite-dimensional L-representation such that $\sigma_V : L \times L \to \mathbb{K}$ is non-degenerate. Then L is reductive.*

Proof. Assume that V is not irreducible, and let us try to produce from V an irreducible representation for which the associated bilinear form is non-degenerate: Let U be a subrepresentation of V. From the exercise above we know that $\sigma_V = \sigma_U + \sigma_{V/U}$. Now σ_V can be non-degenerate if and only if σ_U and $\sigma_{V/U}$ is non-degenerate (Note that it is an orthogonal decomposition.)

If the representation U is irreducible then we are done. If it is reducible, then we repeat the process. Since V is by assumption finite-dimensional, we will eventually arrive at an irreducible representation with non-degenerate associated bilinear form. Thus we can assume without loss of generality that (V, ρ) is non-degenerate.

It follows from Lemma 3.14 that there exists a $\lambda : \mathrm{rad}(L) \to \mathbb{K}$ such that, for any $x \in \mathrm{rad}(L)$, $y \in L$, $v \in V$, we have

$$[x, y]v = xyv - yxv = \lambda(x)yv - y(\lambda(x)v) = 0.$$

Thus we see that

$$\sigma_V([x,y],z) = \mathrm{tr}\,(\rho([x,y])\rho(z)) = 0, \quad \text{for all} \quad x \in \mathrm{rad}(L),\ y,z \in L.$$

In other words

$$\sigma_V([\mathrm{rad}(L), L], L) = 0.$$

But by assumption σ_V is non-degenerate, so we must have that

$$[\mathrm{rad}L, L] = 0.$$

This identity in turn shows us that $\mathrm{rad}(L) \subseteq Z(L)$, which since the opposite inclusion is always true, tells us that $\mathrm{rad}(L) = Z(L)$. Thus L is indeed reductive. □

4.3. **Cartan's Criterion.** In this section, we will work over the complex numbers $\mathbb{C}$ so that we can take advantage of the following result:

Theorem 4.20 (Jordan decomposition)**.** *Any linear transformation x of a complex vector space V has a unique decomposition*

$$x = d + n$$

where d is a diagonalisable linear operator, and n is a nilpotent linear operator, and d and n commute. We call this decomposition the ***Jordan decomposition*** *of x.*

Corollary 4.21. *Let x have Jordan decomposition $x = d + n$.*

1. *there exists a polynomial $p \in \mathbb{C}[X]$, such that $p(x) = d$,*
2. *Fix a basis of V with respect to which d is diagonalisable. Let $\overline{d}$ denote the linear map whose matrix (with respect to this choice of basis) is the complex conjugate of the matrix of d. There is a polynomial $q \in \mathbb{C}[X]$ such that $q(x) = \overline{d}$.*

Exercise 4.22. Let V be a complex vector space and let $L \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra. Use Lie's theorem to show that there exists a basis of V such that $[L, L]$ can be written as strictly upper triangular matrices. Then conclude that $\mathrm{tr}(xy) = 0$, for all $x \in L$, and $y \in [L, L]$.

Proposition 4.23. *Let V be a complex vector space and let L be a Lie subalgebra of $\mathfrak{gl}(V)$. If $\mathrm{tr}(xy) = 0$, for all $x, y \in L$, then L is solvable.*

Proof. We shall show that every $x \in [L, L]$ is a nilpotent linear map. It will then follow from Engel's theorem that every element of $[L, L]$ can be written as a strictly upper triangular matrix, and hence $[L, L]$ is a nilpotent Lie algebra. We can then conclude that L is solvable.

Let $x \in [L, L]$ have Jordan decomposition $x = d + n$, where d is diagonalisable, n is nilpotent, and d and n commute. We can fix a basis of V in which d is diagonal and n is strictly upper triangular. Since our aim is to show that $d = 0$, it will suffice to show that

$$\sum_{i=1}^{m} \lambda_i \overline{\lambda_i} = 0,$$

where λ_i are the diagonal entries of d.

The matrix $\overline{d}$ is diagonal, with diagonal entries $\overline{\lambda}_i$, for $i = 1, \ldots, m$. A simple computation shows that

$$\operatorname{tr}(\overline{d}x) = \sum_{i=1}^{m} \lambda_i \overline{\lambda}_i.$$

Now as $x \in [L, L]$, we may express x as a linear combination of elements of the form $[y, z]$, for $y, z \in L$. Thus it suffices to show that $\operatorname{tr}(\overline{d}[y, z]) = 0$. But this is equivalent to showing

$$\operatorname{tr}([\overline{d}, y]z) = 0.$$

Thus by our hypothesis it suffices to show that $[\overline{d}, y] \in L$. In other words that $\mathrm{ad}_{\overline{d}}$ maps L into L.

Exercise: Show that the Jordan decomposition of the linear operator ad_x is in fact given by

$$\mathrm{ad}_x = \mathrm{ad}_d + \mathrm{ad}_n.$$

From this exercise, and the above corollary to the Jordan decomposition theorem, it follows that there exist a polynomial $q \in \mathbb{C}[X]$ such that

$$q(\mathrm{ad}_x) = \overline{\mathrm{ad}_d} = \mathrm{ad}_{\overline{d}}.$$

Now ad_x maps L into itself, hence any polynomial in ad_x maps L into itself, hence $\mathrm{ad}_{\overline{d}}$ maps L into itself. □

Theorem 4.24. *A Lie algebra L is solvable if and only if $\kappa(L, [L, L]) = 0$.*

Proof. Assume that $\kappa(L, [L, L]) = 0$. Consider the adjoint map $\mathrm{ad} : L \to \mathfrak{gl}(L)$ and its image $\mathrm{ad}(L) \subseteq \mathfrak{gl}(L)$, which is of course again a Lie algebra. For any $x, y, z \in L$, we have

$$\operatorname{tr}(\mathrm{ad}_x[\mathrm{ad}_y, \mathrm{ad}_z]) = \operatorname{tr}(\mathrm{ad}_x, \mathrm{ad}_{[y,z]}) = \kappa(x, [y, z]) = 0.$$

This implies that the conditions of the proposition above are satisfied, and hence that $[\mathrm{ad}(L), \mathrm{ad}(L)]$ is solvable, and hence that $\mathrm{ad}(L)$ is solvable. Recalling that $\ker(\mathrm{ad}) = Z(L)$, we see that $\ker(\mathrm{ad})$ is abelian and hence solvable. Thus since we just showed that $L/Z(L) \simeq \mathrm{ad}(L)$ is solvable, it must hold that L is solvable.

Conversely, assume that L is solvable. Then

$$\mathrm{ad}(L) \simeq L/Z(L)$$

is solvable (since $Z(L)$ and L are). It now follows from Exercise 4.22 that

$$0 = \operatorname{tr}(\mathrm{ad}_x[\mathrm{ad}_y, \mathrm{ad}_z]) = \operatorname{tr}(\mathrm{ad}_x[\mathrm{ad}_{[y,z]}]) = \kappa(x, [y, z]),$$

and hence that $\kappa(L, [L, L]) = 0$ as claimed. □

Lemma 4.25. *Let I be an ideal of a Lie algebra. Then the Killing form κ_I of I coincides with the restriction of the Killing form κ_L to I.*

Proof. If $A \in \mathrm{End}(L)$ satisfies $A(L) \subseteq I$, then $\operatorname{tr}(A) = \operatorname{tr}(A|_I)$. (You should convince yourself that this is true if it is not clear.) In particular, for any $x, y \in I$ the operators ad_x, ad_y map L to I. Applying the first observation to $A = \mathrm{ad}_x \mathrm{ad}_y$, we obtain

$$\kappa_L(x, y) = \operatorname{tr}(\mathrm{ad}_x \mathrm{ad}_y) = \operatorname{tr}(A) = \operatorname{tr}(A|_I) = \kappa_I(x, y).$$

□

Definition 4.26. For two Lie algebras $(L, [-,-]_L)$ and $(K, [-,-]_K)$ over $\mathbb{K}$, their **direct sum** is $(L \oplus K, [-,-])$, where $L \oplus K$ is the direct sum of the vector spaces L and K, and $[-,-]$ is defined by

$$[-,-] : K \oplus L \to \mathbb{K}, \qquad (l_1 \oplus k_1, l_2 \oplus k_2) \mapsto [l_1, l_2]_L + [k_1, k_2]_K.$$

Note that by construction, L and K are ideals of $L \oplus K$.

Theorem 4.27. *Let L be a Lie algebra. The following are equivalent*

1. *L is semisimple,*
2. *the Killing form is non-degenerate (Cartan's criterion),*
3. *L is isomorphic to a direct sum $I_1 \oplus \cdots \oplus I_m$ of simple ideals,*
4. *L is isomorphic to a direct sum $L_1 \oplus \cdots \oplus L_m$ of simple Lie algebras.*

Proof.
(1) → (2) Let L be a semisimple Lie algebra. Consider

$$I := L^{\perp} := \{x \in L \,|\, \kappa(x, L) = 0\},$$

which is an ideal by L-invariance of κ. By the lemma above, we have that $\kappa_I = \kappa|_I = 0$. By Cartan's criterion for solvability, this means that I is solvable. But L is semisimple by assumption, so we must have that $I = 0$. Hence κ is non-degenerate.

(2) → (1) We saw earlier that if κ is non-degenerate, then L is reductive, hence $\mathrm{rad}(L) = Z(L)$. But if $x \in Z(L)$, then $\mathrm{ad}_x = 0$, hence

$$\kappa(x, y) = \mathrm{tr}(\mathrm{ad}_x, \mathrm{ad}_y) = 0, \qquad \text{for all } y \in L.$$

This implies that $x = 0$, as κ is nondegenerate. We conclude that

$$Z(L) = \mathrm{rad}(L) = 0,$$

which is to say, L is semisimple.

(1) → (3) If L is simple then we are done. Assume that $I \subseteq L$ is a non-trivial ideal. Then

$$I^{\perp} := \{x \in L \,|\, \kappa(x, I) = 0\},$$

is an ideal (check that this follows from L-invariance of the Killing form). The Killing form is zero on the ideal $I \cap I^{\perp}$, meaning that it is a solvable ideal.

Since L is semisimple by assumption, it has no non-zero solvable ideals, hence $I \cap I^{\perp} = 0$. This implies that we have a vector space direct sum

$$L = I \oplus I^{\perp}.$$

In fact, $[I, I^{\perp}] \subseteq I \cap I^{\perp} = 0$, meaning that we have a direct sum of Lie algebras.

Exercise: Prove that any ideal I of a semisimple Lie algebra is again semisimple.

From this exercise we see that I and $I^{\perp}$ are semisimple. Following an inductive argument we can now decompose I and $I^{\perp}$ into a direct sum of simple ideals.

(4) → (1) Let $L = L_1 \oplus \cdots \oplus L_n$ be a direct sum of simple Lie algebras. Let $I \subseteq L$ be an abelian ideal. The projection

$$\pi_k : L \to L_k, \qquad (l_1, \ldots, l_n) \mapsto l_n$$

is clearly a surjective Lie algebra map. Hence the the image of I under π_k is an abelian ideal. However, the only abelian ideal of L_k is zero. This implies that $\pi_k(I) = 0$, for all $1 \leq k \leq n$. Thus $I = 0$, implying that L is semisimple. □

Theorem 4.28. *Let L be a semisimple Lie algebra. Then the adjoint map* $\mathrm{ad} : L \to \mathrm{Der}(L)$ *is an isomorphism.*

Proof. We know that $\ker(\mathrm{ad}) = Z(L)$ (if you don't recall the proof, you should try to prove it again). This is an abelian ideal, so since L is by assumption semisimple,

$$\ker(\mathrm{ad}) = Z(L) = 0$$

and hence that ad is an injective map.

Recall that ad maps L into the derivations of L. For any $\delta \in \mathrm{Der}(L)$, and for any $x \in L$, we have

$$[\delta, \mathrm{ad}_x] = \mathrm{ad}_{\delta(x)} \in \mathrm{ad}(L).$$

Thus we see that $\mathrm{ad}(L)$ is an ideal of $\mathrm{Der}(L)$.

The subspace

$$I := \mathrm{ad}(L)^{\perp} = \{x \in \mathrm{Der}(L) \,|\, \kappa_{\mathrm{Der}(L)}(x, \mathrm{ad}(L)) = 0\}$$

is also an ideal. Since $\mathrm{ad}(L) \simeq L$, it is semisimple, we know that

$$\kappa_{\mathrm{ad}(L)} = \kappa_{\mathrm{Der}(L)}|_{\mathrm{ad}(L)}$$

is non-degenerate, and therefore $I \cap \mathrm{ad}(L) = 0$. This implies that $\mathrm{Der}(L) = \mathrm{ad}(L) \oplus I$.

Let $\delta \in I$, and $x \in L$, then we have

$$\mathrm{ad}_{\delta(x)} = [\delta, \mathrm{ad}_x] \in [I, \mathrm{ad}(L)] \subseteq I \cap \mathrm{ad}(L) = 0.$$

But ad is an injective map, hence we must have $\delta(x) = 0$, for all $x \in L$, which is to say $\delta = 0$. Thus, since δ was an arbitrary element of I, we must have that $I = 0$, meaning that $\mathrm{ad}(L) = \mathrm{Der}(L)$. □

Finally, we finish with an alternative characterisation of reductivity similar to the characterisation of semisimplicity given above. We omit the proof which is similar.

Theorem 4.29. *A finite-dimensional complex Lie algebra is reducitive if and only if it is a direct sum of semisimple Lie algebra and an abelian Lie algebra.*

5. Some More Advanced Results

5.1. Semi-Direct Products and Levi Decomposition.

Proposition 5.1. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two Lie algebras, and $\rho : \mathfrak{a} \longrightarrow \mathrm{Der}(\mathfrak{b})$ a Lie algebra homomorphism to the derivations on $\mathfrak{b}$. The vector space $\mathfrak{a} \oplus \mathfrak{b}$ admits a unique Lie bracket restricting to the Lie brackets on $\mathfrak{a}$ and $\mathfrak{b}$ and such that*

$$[a, b] = \rho(a)(b), \qquad \textit{for all } a \in \mathfrak{a},\ b \in \mathfrak{b}.$$

Proof. Exercise. □

Definition 5.2. This is called the **semidirect product** of $\mathfrak{g}$ and $\mathfrak{a}$ and we denote it by $\mathfrak{g} \oplus_\rho \mathfrak{a}$.

Theorem 5.3 (Levi Decomposition)**.** *Let L be a finite-dimensional Lie algebra over a field $\mathbb{K}$ of characteristic zero. Then there exists a semisimple subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$ such that $\mathfrak{g}$ is the semidirect product*

$$\mathfrak{g} \simeq \mathfrak{s} \oplus_\pi \mathrm{rad}(\mathfrak{g}),$$

for a suitable homomorphism $\pi : \mathfrak{s} \to \mathrm{Der}_{\mathbb{K}}(\mathrm{rad}(\mathfrak{g}))$.

5.2. **Ado's Theorem.**

Theorem 5.4 (Ado's Theorem)**.** *Every finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic zero admits a faithful representation representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, for some finite-dimensional V.*

6. The Root Space Decomposition of a Complex Semisimple Lie Algebra

6.1. **The Root System of $\mathfrak{sl}_3$.**

For $\mathfrak{sl}_3$, denote by $\mathfrak{h} \subset \mathfrak{sl}_3$ the 2-dimensional Lie subalgebra of diagonal metrices. Suppose that $h \in \mathfrak{h}$, and denote

$$h := \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

Then we see that, for $i \neq j$,

$$\mathrm{ad}_h(e_{ij}) = (a_i - a_j)e_{ij},$$

which is to say, e_{ij}, for $i \neq j$, is a common eigenvector of ad_h, for all $h \in \mathfrak{h}$. Moreover, since $\mathfrak{h}$ is abelian, $\mathfrak{h} \subseteq \ker(\mathrm{ad}_h)$, for all $h \in \mathfrak{h}$.

We can write this more formally in terms of weights. Defining a function

$$\varepsilon_i : \mathfrak{h} \to \mathbb{C}, \qquad h \mapsto a_i, \qquad h \mapsto a_i,$$

we have

$$\mathrm{ad}_h(e_{ij}) = (\varepsilon_i - \varepsilon_j)(h)e_{ij}.$$

We call the linear functionals $\varepsilon_i - e_j \in \mathfrak{h}^*$, where $\mathfrak{h}^*$ is the $\mathbb{C}$-linear dual of $\mathfrak{h}$, *weights*.

In fact, denoting

$$L_{ij} := L_{\varepsilon_i - \varepsilon_j} := \{x \in \mathfrak{sl}_3(\mathbb{C}) \mid \mathrm{ad}_h(x) = (\varepsilon_i - \varepsilon_j)(h)x, \text{ for all } h \in \mathfrak{h}\},$$

we have that

$$L_{ij} = \mathbb{C}e_{ij}, \qquad \text{for } i \neq j.$$

Thus we see that

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{i \neq j} L_{ij}.$$

6.2. **General Weight Space Decompositions.**

Lemma 6.1. *Let H be an abelian sub-algebra of a Lie algebra L, consisting of semisimple elements, then L admits a common eigenbasis for the operators*

$$\{\mathrm{ad}_h \mid h \in H\}.$$

Given a common eigenvector $x \in L$, the eigenvalues can be encoded by the *weight functional*

$$\alpha_x : H \to \mathbb{C},$$

determined by

$$\mathrm{ad}_h(x) = \alpha(h)x, \qquad \text{for all } h \in \mathfrak{h}.$$

Definition 6.2. For each $\alpha \in \mathfrak{h}^*$, we call

$$L_\alpha := \{x \in L \mid \mathrm{ad}_h(x) = \alpha(h)(x), \quad \text{for all } h \in \mathfrak{h}\},$$

the *weight space* of α.

Note that since

$$L_0 = \{x \in L \mid \mathrm{ad}_h(x) = 0, \text{ for all } h \in \mathfrak{h}\}$$

is the centraliser of $\mathfrak{h}$ in L. Since $\mathfrak{h}$ is abelian, we have that $\mathfrak{h} \subseteq L_0$. In fact, this inclusion is an equality, even though we will not have time to prove it in this course.

Definition 6.3. We denote by Δ the set of non-zero $\alpha \in \mathfrak{h}^*$, for which $L_\alpha \neq 0$.

Lemma 6.4. *The set Δ is finite.*

Proof. Note that we have the decomposition

$$L \simeq L_0 \oplus \bigoplus_{\alpha \in \Delta} L_\alpha.$$

Since each L_α has dimension at least one, it is clear that Δ must be a finite set. □

Lemma 6.5. *For any α, $\beta \in \mathfrak{h}^*$, it holds that*

1. $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$,
2. *if $\alpha \neq -\beta$, then $\kappa(L_\alpha, L_\beta) = 0$,*
3. *the restriction $\kappa : L_0 \times L_0 \to \mathbb{C}$ of the Killing form to L_0 is non-degenerate.*

Proof. 1. Take $x \in L_0$, $y \in L_\beta$, we need to show that if $[x, y]$ is non-zero, then it is an eigenvector of ad_h, for all $h \in \mathfrak{h}$, with eigenvalue $\alpha(h) + \beta(h)$. From the Jacobi identity, we have

$$\begin{aligned} \mathrm{ad}_h([x, y]) = [h, [x, y]] &= [[h, x], y] + [x, [h, y]] \\ &= [\alpha(h)x, y] + [x, \beta(h)y] \\ &= (\alpha + \beta)(h)[x, y]. \end{aligned}$$

2. Since $\alpha+\beta \neq 0$, there exists a $h \in \mathfrak{h}$ such that $(\alpha+\beta)(h) \neq 0$. Now, for any $x \in L_\alpha$ and $y \in L_\beta$, we have

$$\begin{aligned}\alpha(h)\kappa(x,y) &= \kappa([h,x],y)\\ &= -\kappa([x,h],y)\\ &= -\beta(h)\kappa(x,y).\end{aligned}$$

Hence

$$(\alpha+\beta)(h)\kappa(x,y) = 0.$$

Since by assumption $(\alpha+\beta)(h) \neq 0$, we must have that $\kappa(x,y) = 0$.

3. Consider some $z \in L_0$ and $\kappa(z,x) = 0$, for all $x \in L_0$. By 2, we know that L_0 is perpendicular to L_α (with respect to the Killing form), for all $\alpha \neq 0$. If $a \in L$, then we can write x as

$$x = x_0 + \sum_{\alpha\in\Delta} x_\alpha, \qquad \text{for } x_\alpha \in L_\alpha.$$

By linearity $\kappa(z,x) = 0$, for all $x \in L$. Since κ in non-degenerate, it follows that $z = 0$. □

6.3. **Cartan Subalgebras.**

Definition 6.6. An element h of a Lie algebra L is called *semisimple* if ad_h is diagonalisable, that is to say, if L admits an eigenbasis for the operator ad_h.

Definition 6.7. A Lie subalgebra $\mathfrak{h}$ of a Lie algebra L is said to be a *Cartan subalgebra* if $\mathfrak{h}$ is abelian, every element of $\mathfrak{h}$ is semisimple, and $\mathfrak{h}$ is maximal with respect to these properties.

Since the Killing form $\kappa : L_0 \times L_0 \to \mathbb{C}$ is non-degenerate, and L_0 is finite-dimensional, we have an induced isomorphism

$$\flat : \mathfrak{h} \to \mathfrak{h}^*, \qquad h \mapsto \kappa(h,-).$$

We denote the inverse to $\flat$ by

$$\sharp : \mathfrak{h}^* \to \mathfrak{h}.$$

By abuse of notation, we denote the induced bilinear form on $\mathfrak{h}^*$ by $(-.-)$. Explicitly

$$(\alpha,\beta) = (\alpha^\sharp,\beta^\sharp), \qquad \text{for } \alpha,\beta \in \mathfrak{h}.$$

Lemma 6.8. *Let* $\alpha \in \Delta$.

1. *If* $x \in L_\alpha$, *and* $y \in L_{-\alpha}$, *then*
$$[x,y] = \kappa(x,y)\alpha^\flat,$$
2. $(\alpha,\alpha) \neq 0$,
3. *Let* $x_\alpha \in L_\alpha$, *and* $y_\alpha \in L_{-\alpha}$, *chosen such that*
$$\kappa(x_\alpha,y_\alpha) = \frac{2}{(\alpha,\alpha)}.$$

Then, denoting

$$h_\alpha := \frac{2\alpha^\flat}{(\alpha,\alpha)},$$

it holds that $\alpha(h_\alpha) = 2$*, and the elements*

$$\{x_\alpha, y_\alpha, h_\alpha\}$$

form a Lie subalgebra of $\mathfrak{g}$ *isomorphic to* $\mathfrak{sl}_2$*. We denote this copy of* $\mathfrak{sl}_2$ *by* $(\mathfrak{sl}_2)_\alpha$*.*

Proof. 1. For any $h \in \mathfrak{h}$, it holds that

$$\begin{aligned}\kappa([x,y],h) &= \kappa(x,[y,h]) \\ &= -\kappa(x,[h,y]) \\ &= \alpha(h)\kappa(x,y) \\ &= \kappa(\alpha^\flat, h)\kappa(x,y) \\ &= \kappa(\kappa(x,y)\alpha^\flat, h).\end{aligned}$$

Since $[x,y] \in \mathfrak{h}$, and $\kappa(x,y)\alpha^\flat \in \mathfrak{h}$, it follows from non-degeneracy of the Killing form that $[x,y] = (x,y)\alpha^\flat$.

2. Assume that

$$\kappa(\alpha,\alpha) = \alpha^\flat(\alpha) = 0.$$

Let $0 \neq x \in L_\alpha$. Since the Killing form is non-degenerate on $L_\alpha \times L_{-\alpha}$, there exists a $y \in L_{-\alpha}$, such that $\kappa(x,y) \neq 0$. Then

$$[x,y] = \kappa(x,y)\alpha^\flat \neq 0.$$

Consider now the subspace

$$V := \mathrm{span}_{\mathbb{C}}\{x, \ , y, \ [x,y]\}.$$

We have

$$[h,x] = \alpha(h)x = \kappa(\alpha, \kappa(x,y)\alpha^\flat)x = \kappa(x,y)\kappa(\alpha,\alpha) = 0.$$

Similarly, $[h,y] = 0$. This implies that V is a solvable Lie algebra. By Lie's theorem, there exists a basis of L such that V acts as upper-triangular matrices. This implies that

$$\mathrm{ad}_h = \mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]$$

acts as a strictly upper triangular matrix, hence it is a nilpotent operator. However, since $h \in \mathfrak{h}$, it is a semisimple operator, implying that $h = 0$. To avoid contradiction, we must conclude that $(\alpha,\alpha) = 0$.

3. We have that

$$\kappa(\alpha, h_\alpha) = \frac{2}{\kappa(\alpha,\alpha)}\kappa(\alpha,\alpha) = 2.$$

Moreover, we have

$$[x_\alpha, y_\alpha] = \kappa(x_\alpha, y_\alpha)\alpha^\flat = \frac{2\alpha^\flat}{(\alpha,\alpha)} = h_\alpha.$$

Finally, we see that

$$[h_\alpha, x_\alpha] = \alpha(h_\alpha)x_\alpha = \alpha\left(\frac{2\alpha^\flat}{(\alpha,\alpha)}\right)x_\alpha = \frac{2(\alpha,\alpha)}{(\alpha,\alpha)}x_\alpha = 2x_\alpha.$$

Thus we see that we do indeed have a Lie subalgebra isomorphic to $\mathfrak{sl}_2$. □

Theorem 6.9. *It holds that*

1. Δ *spans* $\mathfrak{h}^*$,
2. *the vector space*
$$V_\alpha := \mathbb{C}h_\alpha \bigoplus_{k\in\mathbb{Z}\setminus\{0\}} K_{k\alpha}$$
is an irreducible representation of $(\mathfrak{sl}_2)_\alpha$.
3. *For all* $\alpha \in \Delta$, *the weight space* L_α *is one-dimensional.*
4. *For any* $\alpha \in \Delta$, *the reflection*
$$s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*, \qquad \lambda \mapsto \lambda - 2\frac{(\alpha,\lambda)}{(\alpha,\alpha)},$$
leaves the hyperplane
$$\alpha^\perp = \{\lambda \in \mathfrak{h}^* \mid (\alpha,\lambda) = 0\}$$
invariant, and maps α *to* $-\alpha$. *Hence* $s_\alpha(\Delta) \subseteq \Delta$.
5. *If* α, $\beta \in \Delta$, *then it holds that*
$$\beta(h_\alpha) = \frac{2\kappa(\alpha,\beta)}{\kappa(\alpha,\alpha)} \in \mathbb{Z}.$$
6. *For any* $\alpha \in \Delta$, *and* $c \in \mathbb{C}$, *we have that* $c\alpha \in \Delta$ *if and only if* $c = \pm 1$.
7. *For any* α, $\beta \in \Delta$, *such that* $\beta \neq \pm\alpha$, *we have that*
$$\bigoplus_{k\in\mathbb{Z}} L_{\beta\alpha+k\alpha}$$
is an irreducible $(\mathfrak{sl}_2)_\alpha$*-submodule of* $\mathfrak{g}$.

For any two roots $\alpha,\ \beta \in \Delta$, *such that* $\alpha + \beta \in \Delta$, *we have*

$$[L_\alpha, L_\beta] = L_{\alpha+\beta}.$$

7. Root Systems

7.1. Definition of a Root System.

Definition 7.1. Let E be a real vector space endowed with an inner product $(-,-)$. A **root system** is a finite subset $\Delta \subseteq E\setminus\{0\}$, whose elements are called **roots**, such that

1. Δ spans E as a real vector space,
2. for any α, $\beta \in \Delta$, the real number $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer,

3. for any $\alpha \in \Delta$, the associated *reflection*

$$s_\alpha : E \to E, \qquad \lambda \mapsto \lambda - \frac{2(\alpha,\lambda)}{\alpha,\alpha},$$

satisfies $s_\alpha(\Delta) \subseteq \Delta$,

4. for any $\alpha \in \Delta$, the only multiples of α which are also roots are $\pm\alpha$.

Remark 7.2. Note first that $s_\alpha(\alpha) = -\alpha$. Moreover, if α and λ are orthogonal, that is if $(\alpha,\lambda) = 0$, then we have $s_\alpha(\lambda) = \lambda$. Thus we can interpret s_α as reflection with respect to the hyperplane

$$H_\alpha := \alpha^\perp = \{\lambda \in E \mid (\alpha,\lambda) = 0\}.$$

We also note that

$$s_\alpha^2 = \mathrm{id}.$$

Definition 7.3. The *Weyl group* W of the root system $R \subseteq E$ is the subgroup of $\mathrm{GL}(E)$ generated by reflections s_α, for $\alpha \in R$.

Lemma 7.4. *The Weyl group is a finite subgroup of* $\mathrm{O}(E)$*, the orthogonal linear maps on* E*. Moreover,* W *maps* R *to itself.*

Proof. Every s_α is orthogonal linear map since

$$\begin{aligned}(s_\alpha(\lambda), s_\alpha(\mu)) &= \left(\lambda - \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}\alpha,\ \mu - \frac{2(\alpha,\mu)}{(\alpha,\alpha)}\alpha\right)\\ &= (\lambda,\mu) - \frac{4(\alpha,\lambda)(\alpha,\mu)}{(\alpha,\alpha)} + \frac{4(\alpha,\lambda)(\alpha,\mu)}{(\alpha,\alpha)^2}(\alpha,\alpha)\\ &= (\lambda,\mu).\end{aligned}$$

This we see that $W \subseteq O(E)$.

Since we already know that $s_\alpha(R) = R$, it is clear that W maps R to R. Now if some $w \in W$ acts trivially on R, then w acts trivially on E since R spans E. Thus the mapping from W to $\mathrm{Aut}(R)$, the group of permutations of R, has trivial kernel, which is to say, it is an injective map. Since R is a finite set, $\mathrm{Aut}(R)$ is also finite, and so, W must be finite. ☐

Example 7.5. Let E be the vector space $\{x \in \mathbb{R}^{n+1} \mid \sum_i x_i = 0\}$, endowed with the restriction of the usual inner product of $\mathbb{R}^{n+1}$ to E. As a root space we propose

$$\Delta := \{\alpha_{ij} = e_i - e_j \mid i \neq j\}.$$

This set spans E. Moreover, i, j, k distinct,

$$(\alpha_{ij},\alpha_{ij}) = 2, \quad (\alpha_{ij},\alpha_{jl}) = -1, \quad (\alpha_{ij},\alpha_{ki}) = -1$$

we see that $\frac{2(\alpha,\beta)}{(\beta,\beta)}$ is an integer.

Finally, we see that

$$s_{\alpha_{ij}}(\lambda) = \lambda - \frac{2(\alpha_{ij},\lambda)}{(\alpha_{ij},\alpha_{ij})}\alpha_{ij} = \lambda - (e_i - e_j,\lambda)(e_i - e_j).$$

Explicitly this means that

$$s_{\alpha_{ij}}(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_{n+1}) = (\lambda_1, \ldots, \lambda_j, \ldots, \lambda_i, \ldots, \lambda_{n+1}).$$

This implies that $s_\alpha(\Delta) \subseteq \Delta$. We denote this root system by A_n.

7.2. **Some Basic Results.**

Theorem 7.6. *Let $\alpha, \beta \in \Delta$ be two roots such that $\alpha \neq \pm\beta$ and, without loss of generality, let us assume that $\|\beta\| \leq \|\alpha\|$. Let θ be the angle between α and β. Then we have only the following possibilities:*

	(β, α)	(α, β)	θ
1	0	0	$\pi/2$
2	1	1	$\pi/3$
3	-1	-1	$2\pi/3$
4	1	2	$\pi/4$
5	-1	-2	$3\pi/4$
6	1	3	$\pi/6$
7	-1	-3	$5\pi/6$

Proof. Consider the two integers $m = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ and $n = \frac{2(\alpha,\beta)}{(\beta,\beta)}$. Since $(\alpha, \beta) = \|\alpha\|\|\beta\|\cos(\theta)$, we have that

$$mn = 4\frac{(\alpha, \beta)^2}{\|\alpha\|^2\|\beta\|^2} = 4\cos^2(\theta).$$

Since by assumption $\beta \neq \pm\alpha$, we have $\cos(\theta) \neq \pm 1$. Hence $mn \in \{0, 1, 2, 3\}$, and

$$\frac{m}{n} = \frac{(\beta, \beta)}{(\alpha, \alpha)} = \frac{\|\beta\|}{\|\alpha\|} \leq 1,$$

hence $|m| \leq |n|$. If $m = 0$, then $n = 0$. If $m = \pm 1$, then $n = \pm 1, \pm 2, \pm 3$. The angle θ is determined from the identity $4\cos^2(\theta) = mn$. □

Corollary 7.7. *Let $\alpha, \beta \in \Delta$. Then*

1. *if $(\alpha, \beta) > 0$, then $\alpha - \beta \in \Delta$,*
2. *if $(\alpha, \beta) < 0$, then $\alpha + \beta \in \Delta$.*

Proof. Denote $m := \langle \alpha, \beta^\vee \rangle > 0$, $n := \langle \beta, \alpha^\vee \rangle$. Then $m = 1$ or $n = 1$. If $m = 1$, then

$$s_\beta(\alpha) = \alpha - (\alpha, \beta^\vee)\beta = \alpha - \beta \in \Delta.$$

If $m = 1$, then

$$s_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha = \beta - \alpha \in \Delta,$$

hence $\alpha - \beta \in \Delta$. Finally, we observe that the second implication follows from the first. □

7.3. **Bases of Root Systems.**

Definition 7.8. A subset $\Pi \subseteq \Delta$ is called a **base** of Δ if

1. Π is a basis of E,
2. for any root $\beta = \sum_{\alpha \in \Pi} n_\alpha \alpha$, either

$$\{n_\alpha \,|\, \alpha \in \Pi\} \subseteq \mathbb{Z}_{\geq 0}, \qquad \text{or} \qquad \{n_\alpha \,|\, \alpha \in \Pi\} \subseteq \mathbb{Z}_{\leq 0},$$

We call the elements of a base **simple** roots.

Definition 7.9. The **positive root system** is the set

$$\Delta_+ := \{\beta = \sum_{\alpha \in \Pi} n_\alpha \alpha \in \Delta \,|\, n_\alpha \geq 0, \text{ for all } \alpha \in \Pi\}.$$

The **negative root system** is the set $-\Delta_+$.

We note that we have $\Delta = \Delta_+ \cup \Delta_+$.
One can construct bases of Δ as follows. Let

$$H_\alpha := \{\lambda \in E \,|\, (\alpha, \lambda) = 0\},$$

for any root α. We call any element

$$\gamma \in E \backslash \cup_{\alpha \in \Delta} H_\alpha,$$

a *regular vector*. Define

$$\Delta_+(\gamma) := \{\alpha \in \Delta \,|\, (\alpha, \gamma) > 0\}, \qquad \Delta_-(\gamma) = \{\alpha \in \Delta \,|\, (\alpha, \gamma) < 0\}.$$

Then $\Delta_-(\gamma) = -\Delta_+(\gamma)$ and $\Delta = \Delta_+(\gamma) \cup \Delta_-(\gamma)$. The sets $\Delta_+(\gamma)$ and $\Delta_-(\gamma)$ are called the sets of positive and negative roots respectively.

Definition 7.10. We say that an element $\alpha \in \Delta_+(\gamma)$ is a *simple root* if it cannot be written as a sum of two positive roots. Denote the set of all simple roots by $\Pi(\gamma)$.

Lemma 7.11. *Any positive root can be written as a sum of simple roots.*

Proof. If $\alpha \in \Delta_+(\gamma)$ is not simple, then $\alpha = \alpha' + \alpha''$ for some $\alpha', \alpha'' \in \Delta_+(\gamma)$. This implies that $(\alpha', \gamma) < (\alpha, \gamma)$ and $(\alpha''.\gamma) < (\alpha, \gamma)$. Applying an inductive argument on (α, γ), we obtain representations of α' and α'' as sums of simple roots. Thus we can represent α as a sum of simple roots. □

Theorem 7.12. *For any regular $\gamma \in E$, the set of simple roots $\Pi(\gamma)$ is a base of Δ. Any base of the root system Δ has this form.*

Example 7.13. Returning to the root system A_n, we choose

$$\gamma := (n, n-1, ..., 1) \in \mathbb{R}^n.$$

Then we see that

$$(\gamma, \alpha_{ij}) = (\gamma, e_i) - (\gamma, e_j) = (n+1-i) - (n+1-j) = j - i > 0$$

if and only if $i < j$. Therefore

$$\Delta_+ = \{e_i - e_j \,|\, i < j\}.$$

For any $i < j$, we can write

$$e_i - e_j = (e_i - e_{i+1}) + \cdots + (e_{j-1} - e_j).$$

Therefore the simple roots are

$$\alpha_i = e_i - e_{i+1}, \qquad \text{for } 1 \leq i \leq n-1$$

and

$$\Pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$$

is a base of Δ. Note that Π is a vector space basis of E.

8. Linear Groups

Consider the group of invertible $n \times n$ matrices over $\mathbb{R}$: where the group operation is given by the product of matrices. We call it the general linear group of order n.

Definition 8.1. A linear group is a closed subgroup of $GL_n(\mathbb{R})$ endowed with its operator topology.

Definition 8.2. The *exponential map* is the function

$$\exp : \mathfrak{gl}_n \to \mathrm{GL}_n(\mathbb{R}), \qquad A \mapsto \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

Definition 8.3. Given a linear group $G \subseteq \mathrm{GL}_n(R)$, define

$$L(G) = \{A \in \mathfrak{gl}_n(\mathbb{R} \,|\, e^{tA} \in G, \text{ for all } t \in \mathrm{R}\}.$$

We call $L(G)$ the *Lie algebra* of G.

Theorem 8.4. *For any Linear group G, it holds that $L(G)$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$.*

Example 8.5. The group

$$\mathrm{SL}_n(\mathbb{R}) := \{A \in \mathrm{GL}_n(\mathbb{R}) \,|\, \det(A) = 1\}$$

is called the *special linear group of order n*

Example 8.6. The group

$$O(n) := \{A \in \mathrm{GL}_n(\mathbb{R}) \,|\, A^t A = 1\}$$

is called the *orthogonal group of order n*. The *special orthogonal group* of order n is the subgroup of $O(n)$ whose determinant is 1.

Example 8.7. The *unitary group* is the group

$$U(n) := \{A \in \mathrm{GL}_n(\mathbb{C}) \,|\, A^* A = 1\}, \qquad \text{where } A^* = \overline{A}^t.$$

The *special unitary group* is

$$SU(n) := \{A \in \mathrm{GL}_n(\mathbb{C}) \,|\, A^* A = 1, \det(A) = 1\}.$$

For example

$$U(1) = \{z \in \mathbb{C}^* \,|\, z\overline{z} = 1\} = \{z \in \mathbb{C} \,|\, |z| = 1\} = S^1,$$

where S^1 is the circle in the $\mathbb{R}^2$. Moreover, we see that $\mathrm{SU}(1) = \{1\}$, that is to say the group with the single element 1.

Exercise: Show that the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$ is $\mathfrak{gl}_n(\mathbb{R})$.