LIE ALGEBRAS AND LIE GROUPS 1

Please submit a solution to four of the following five problems for full marks.

(1) Let $\mathbb{K}$ be a field, and V a $\mathbb{K}$-vector space. Let $\mathfrak{gl}(V)$ be the space of $\mathbb{K}$-linear maps from V to itself, endowed with the bilinear map

$$[-,-] : \mathfrak{gl}(V) \times \mathfrak{gl}(V) \to \mathfrak{gl}(V), \qquad [A,B] \mapsto A \circ B - B \circ A.$$

Show that $\mathfrak{gl}(V)$ is a Lie algebra.

(2) Let $\mathbb{K}$ be a field and $A, B \in M_n(\mathbb{K})$, show that

$$\mathrm{tr}(AB) = \mathrm{tr}(BA)$$

(3) For $l \in \mathbb{Z}_{>0}$, we denote by $\mathfrak{sp}(2l, \mathbb{C})$ the Lie algebra associated to bilinear form given by

$$S := \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix},$$

where id is the $l \times l$-identity matrix. Show that a matrix $A \in \mathfrak{gl}_S(2l, \mathbb{C})$ belongs to $\mathfrak{sp}(2l, \mathbb{C})$ if and only if it is of the form

$$\begin{pmatrix} m & p \\ q & -m^t \end{pmatrix},$$

where p and q are symmetric matrices, and conclude from this the dimension of $\mathfrak{sp}(l, \mathbb{C})$.

(4) Let L be a Lie algebra, and $J \subseteq L$ be an ideal of L, and let $I \subseteq J$ be an ideal of J, then I is an ideal of L.

(5) Show that a Lie algebra is solvable if and only if it there exists a filtration by ideals

$$0 = I_n \subset I_{n-1} \subset \cdots \subset I_1 = L$$

such that each quotient Lie algebra I_k / I_{k+1} is abelian.

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