EXAMPLES

1. Basis Examples

Example 1.1. Consider the special case $\mathfrak{sl}_2$ of the general Lie algebra $\mathfrak{sl}_n$. An explicit basis is given by the three matrices:

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The Lie bracket acts according to

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

Note that the matrix product satisfies

$$e.f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \notin \mathfrak{sl}_2.$$

Thus we see that while $\mathfrak{sl}_2$ is closed under the Lie bracket, it is *not* closed under matrix multiplication.

Example 1.2. Define the algebra of upper-triangular matrices by

$$\mathfrak{b}_n(\mathbb{K}) := \{(a_{ij}) \in \mathfrak{gl}_n(\mathbb{K}) \mid a_{ij} = 0, \text{ for } i > j\},$$

the algebra of strictly upper-triangular matrices by

$$\mathfrak{n}_n(\mathbb{K}) := \{(a_{ij}) \in \mathfrak{gl}_n(\mathbb{K}) \mid a_{ij} = 0, \text{ for } i \geq j\},$$

the algebra of diagonal matrices

$$\mathfrak{t}_n(\mathbb{K}) = \{(aij) \in \mathfrak{gl}_n(\mathbb{K}) \mid a_{ij} = 0 \text{ for } i \neq j\}.$$

Note that all of them are Lie subalgebras of $\mathfrak{gl}_n(\mathbb{K})$.

Example 1.3. Let V be a $\mathbb{K}$-vector space of dimension n and $\sigma : V \times V \to \mathbb{K}$ be a bilinear form. Define

$$\mathfrak{o}(V, \sigma) = \{A \in \mathfrak{gl}(V) \mid \sigma(Av, w) = -\sigma(v, Aw), \text{ for all } v, w \in V\}.$$

It is a Lie subalgebra of $\mathfrak{gl}(V)$, called the *orthogonal Lie algebra*. If $A, B \in \mathfrak{o}(V, \sigma)$, then

$$\begin{aligned} \sigma([A, B]v, w) &= \sigma(ABv, w) - \sigma(BAv, w) \\ &= -\sigma(Bv, Aw) + \sigma(Av, Bw) \\ &= \sigma(v, BAw) - \sigma(v, ABw) \\ &= -\sigma(v, [A, B]w), \end{aligned}$$

and therefore $[A, B] \in \mathfrak{o}(V, \sigma)$. If $V = \mathbb{K}^n$, there exists a (unique) matrix S such that

$$\sigma(v, w) = v^t S w, \qquad \text{for all } v, w \in V.$$

The condition $\sigma(Av, w) = -\sigma(v, Aw)$ can be written in the form

$$(Av)^t Sw = v^t A^t Sw = -v^t SAw.$$

As it is satisfied for all $v, w \in V$, we conclude that $A^t S = -SA$. Therefore

$$\mathfrak{o}(V, \sigma) = \mathfrak{o}(n, S) = \{A \in \mathfrak{gl}_n(\mathbb{K}) \mid A^t S = -SA\}.$$

If σ is non-degenerate (or equivalently if S is invertible), we can write $S^{-1}A^t S = -A$. Therefore, we see that

$$-\mathrm{tr}(A) = \mathrm{tr}(S^{-1}ATS) = \mathrm{tr}(SS^{-1}AT) = \mathrm{tr}(A^t) = \mathrm{tr}(A).$$

Hence $\mathrm{tr}(A) = 0$ and $A \in \mathfrak{sl}_n(k)$. For this reason the Lie algebra $\mathfrak{o}(V, \sigma)$ is also denoted by $\mathfrak{so}(V, \sigma)$ and is called the *special orthogonal Lie algebra.*

Example 1.4. If σ is symmetric non-degenerate and $\mathbb{K}$ is an algebraically closed field with $\mathrm{char}(\mathbb{K}) \neq 2$, then one can choose a basis of V such that σ is given by the identity matrix $S = \mathrm{id}$. The algebra

$$\mathfrak{so}(n) = \mathfrak{so}(V, \sigma) = \{A \in \mathfrak{gl}_n(\mathbb{K}) \mid A^t = -A\}$$

consists of skew-symmetric matrices and is called the *(special) orthogonal Lie algebra.*

Example 1.5. Let σ be a skew-symmetric non-degenerate form for a vector space V. One can show that $\dim(V) = 2n$, for some natural number n, and that there exists a basis of V such that σ is given by the matrix

$$\begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}$$

The algebra

$$\mathfrak{sp}(2n, k) = \mathfrak{so}(V, \sigma) = \{A \in \mathfrak{gl}_n(k) \mid A^t S = -SA\}$$

is called the *symplectic Lie algebra.*

Example 1.6. The set

$$\mathfrak{u}_n := \{A \in \mathfrak{gl}_n(\mathbb{C}) \mid A^* = -A\}, \qquad \text{where } A^* := \overline{A}^t$$

is a vector space over $\mathbb{R}$ (but not over $\mathbb{C}$). It is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$ (over $\mathbb{R}$), called the *unitary Lie algebra.* For example

$$\mathfrak{u}_1 = \{x + iy \in \mathbb{C} \mid x - iy = -(x + iy)\} = \{x + iy \in \mathbb{C} \mid x = 0\} = i\mathbb{R}.$$

Define the special unitary Lie algebra to be

$$\mathfrak{su}_n := \mathfrak{u}_n \cap \mathfrak{sl}_n(\mathbb{C}).$$

Example 1.7. Let σ be the symmetric non-degenerate form given by the matrix

$$\begin{pmatrix} \mathrm{id}_p & 0 \\ 0 & -\mathrm{id}_q \end{pmatrix}.$$

The algebra

$$\mathfrak{so}(p, q) = \mathfrak{so}(V, \sigma) = \{A \in \mathfrak{gl}_n(k) \mid A^t S = -SA\}$$

is called the *Lorentzian Lie algebra.*

1.1. **The Witt Algebra.** Consider the algebra of polynomials $\mathbb{C}[t]$. Its Lie algebra of derivations $\mathrm{Der}(\mathbb{C}[t])$ is called the *Witt algebra*.

Let us try to find a concrete description of the Witt algebra. There is an obvious derivation

$$\frac{d}{dt} : \mathbb{C}[t] \to \mathbb{C}[t].$$

More generally, for any $f \in \mathbb{C}[t]$, the map

$$f\frac{\mathrm{d}}{\mathrm{d}t} : \mathbb{C}[t] \to \mathbb{C}[t], \qquad g \mapsto f\frac{\mathrm{d}g}{\mathrm{d}t}$$

is also a derivation. Conversely, given an arbitrary derivation δ, consider the polynomial $f := \delta(t) \in \mathbb{C}[t]$. Then

$$\delta(t^n) = nt^{n-1}\delta(t) = nt^{n-1}f = f\frac{\mathrm{d}}{\mathrm{d}t}(t^n).$$

Therefore $\delta = f\mathrm{d}/\mathrm{d}t$, and we see that $\mathrm{Der}(\mathbb{C}[t]) = \mathbb{C}[t]\mathrm{d}/\mathrm{d}t$. Explicitly, the formula

$$\left[f\frac{\mathrm{d}}{\mathrm{d}t}, g\frac{\mathrm{d}}{\mathrm{d}t}\right] = \left(f\frac{\mathrm{d}g}{\mathrm{d}t} - g\frac{\mathrm{d}f}{\mathrm{d}t}\right)\frac{\mathrm{d}}{\mathrm{d}t}$$

describes the Lie bracket of the Witt algebra.

1.2. **Ideals and Quotients.**

Example 1.8. Consider the Lie algebra $L = \mathfrak{sl}_2(\mathbb{K})$. We know that it has a basis e, h, f with multiplication

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

This implies that

$$L' = [L, L] = L.$$

Let us build on this and show that the only ideals of L are zero and L. Let $I \subseteq L$ be a non-zero ideal and

$$ae + bf + ch \in I, \qquad \text{for some } a, b, c \in \mathbb{K}.$$

a non-zero element. If $a \neq 0$, then we can apply ad_f twice and obtain

$$a[f, [f, e]] = 2af \in I.$$

If $a = 0$ and $b \neq 0$, we apply ad_h and obtain

$$-2bf \in I.$$

If $a = b = 0$ and $c \neq 0$, then we can apply ad_f and obtain

$$2cf \in I.$$

Finally, since $f \in I$ generates the whole of L as an ideal, we must have that $L = I$.

In particular, we see that since $\mathfrak{sl}_2$ is not abelian, and the center $Z(\mathfrak{sl}_2)$ is an ideal, we must have that

$$Z(\mathfrak{sl}_2) = 0.$$